CE 291F: 48-Hour Midterm

Due on April 8, 2009

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Problem 1a

Derive algebraically the Laplace transform of equation (1).

PDEs:

$$D_0 q_{xx} - C_0 q_x = q_t \qquad \forall x \in (0, L), \forall t \in (0, T] \tag{1}$$

$$B_0 z_t + q_x = 0 \qquad \forall x \in (0, L), \forall t \in (0, T] \tag{2}$$

BC:

$$q(L, t) = bz(L, t) \qquad \forall t \in (0, T] \tag{3}$$

Equation (1) can be adjusted as follows, to prepare it for transformation:

$$D_0 q_{xx} - C_0 q_x - q_t = 0$$

We use the Laplace transformation (taken with the time variable) as defined below:

$$\mathcal{L}(F(x,t)) = \hat{F}(x,s)$$

$$\hat{F}(x,s) = \int_0^{+\infty} e^{-st} F(x,t) dt \tag{4}$$

Applying the Laplace transform (from equation (4)) to the PDE and distributing e^{-st} gives:

$$\hat{F}(x,s) = \int_0^{+\infty} (e^{-st} D_0 q_{xx} - e^{-st} C_0 q_x - e^{-st} q_t) dt$$

$$\hat{F}(x,s) = D_0 \int_0^{+\infty} e^{-st} q_{xx} dt + C_0 \int_0^{+\infty} -e^{-st} q_x dt - \int_0^{+\infty} e^{-st} q_t dt$$

Note that the $\int_0^{+\infty} e^{-st} q_t dt$ term requires the use of the chain rule because its derivative depends on t, which results in an s appearing with that term in the Laplace domain.

$$D_0 \hat{q}_{xx}(x,s) - C_0 \hat{q}_x(x,s) - s\hat{q}(x,s) = 0$$

$$D_0 \hat{q}_{xx}(x,s) - C_0 \hat{q}_x(x,s) = s\hat{q}(x,s) \tag{5}$$

Problem 1b

Solve the ODE in equation (5) in the variable x.

i) Show that the equation satisfied by $\lambda(x)$ is $D_0\lambda^2 - C_0\lambda - s = 0$.

First we find the necessary partial derivatives of $\hat{q}(x,s)$:

$$\hat{q}_x(x,s) = \lambda(s)A(s)e^{\lambda(s)x}$$
$$\hat{q}_{xx}(x,s) = \lambda^2(s)A(s)e^{\lambda(s)x}$$

We then plug these two equations into equation (5):

$$D_0(\lambda^2(s)A(s)e^{\lambda(s)x}) - C_0(\lambda(s)A(s)e^{\lambda sx}) = S(A(s)e^{\lambda(s)x})$$

At this point it can be seen that the $A(s)e^{\lambda(s)x}$ terms all cancel out, leaving the following equation for $\lambda(x)$ which we were seeking:

$$D_0\lambda^2 - C_0\lambda - s = 0 \tag{6}$$

ii) Show that the solutions to this equation are $\lambda_{1,2}(s) = \alpha \pm \gamma(s)$ where $\alpha = \frac{C_0}{2D_0}, \gamma(s) = \beta\sqrt{s + \frac{\alpha^2}{\beta^2}}$ and $\beta = \frac{1}{\sqrt{D_0}}$.

Solve for $\lambda_{1,2}(s)$ by applying the quadratic formula to equation (6):

$$\lambda_{1,2} = \frac{C_0}{2D_0} \pm \frac{\sqrt{C_0^2 + 4D_0S}}{2D_0}$$

Set $\alpha = \frac{C_0}{2D_0}$, factor $4D_0$ out of the discriminant, and continue simplification:

$$\lambda_{1,2} = \alpha \pm \frac{1}{2D_0}\sqrt{4D_0(S + (\frac{1}{4D_0})C_0^2)}$$
$$\lambda_{1,2} = \alpha \pm \frac{2\sqrt{D_0}}{2D_0}\sqrt{S + \frac{C_0^2}{4D_0}}$$
$$\lambda_{1,2} = \alpha \pm \frac{1}{\sqrt{D_0}}\sqrt{S + (\frac{C_0^2}{4D_0^2})(D_0)}$$

Finally, making use of $\beta = \frac{1}{\sqrt{D_0}}$ gives the desired result:

$$\lambda_{1,2} = \alpha \pm \beta\sqrt{S + \frac{\alpha^2}{\beta^2}} \tag{7}$$

iii) Conclude by finding the general form of the solution of the ODE:

We begin, as is suggested, by assuming that the solutions are of the form $\hat{q}(x,s) = A(s)e^{\lambda(s)x}$. We are dealing with a homogeneous linear ODE, which, by definition obeys the superposition rule. Thus any linear combination of solutions to the ODE is also a solution. Since the previous exercise (ii) yielded two solutions, λ_1 and λ_2, we write the general form of the solution to the ODE as a linear combination of these two solutions, choosing distinct $A(s)$ coefficients. The result follows:

$$\hat{q}(x,s) = A_1(s)e^{\lambda_1(s)x} + A_2(s)e^{\lambda_2(s)x} \tag{8}$$

Problem 1c

Carry out a Laplace transform of equation (2).

We reference the definition of the Laplace transform in equation (4), plugging equation (2) into it, following the same procedure as was done previously in Problem 1a with equation (1) (again remembering to apply the chain rule to the $\int_0^{+\infty} e^{-st} z_t dt$ term because its derivative depends on t):

$$\begin{aligned}
\hat{F}(x,s) &= \int_0^{+\infty} (B_0 e^{-st} z_t + e^{-st} q_x) dt \\
\hat{F}(x,s) &= B_0 \int_0^{+\infty} e^{-st} z_t dt + \int_0^{+\infty} e^{-st} q_x dt \\
\hat{F}(x,s) &= B_0 s\hat{z}(x,s) + \hat{q}_x(x,s) \qquad (9)
\end{aligned}$$

Finally, taking the partial derivative with respect to x of $\hat{q}(x,s)$ as defined in equation (8) and applying it to equation (9) as $\hat{f}(x,s)$ vanishes gives us the desired expression for $\hat{z}(x,s)$:

$$\begin{aligned}
\hat{q}_x(x,s) &= A_1(s)\lambda_1(s)e^{\lambda_1(s)x} + A_2(s)e^{\lambda_2(s)x} \\
0 &= B_0 s\hat{z}(x,s) + (A_1(s)\lambda_1(s)e^{\lambda_1(s)x} + A_2(s)e^{\lambda_2(s)x}) \\
\hat{z}(x,s) &= \frac{1}{-B_0 s}(A_1(s)\lambda_1(s)e^{\lambda_1(s)x} + A_2(s)e^{\lambda_2(s)x}) \qquad (10)
\end{aligned}$$

Problem 1d

Solve for $A_{1,2}$ using the boundary conditions, given as $\hat{u}(s) = \hat{q}(0, s)$ and $\hat{q}(L, s) = b\hat{z}(L, s)$.

For this problem we use $\hat{q}(x, s)$ and $\hat{z}(x, s)$ as defined in equations (8) and (10), respectively. We begin by setting $A_2(s) = \hat{u}(s) - A_1(s)$ according to the first boundary condition. We then plug this expression into the second boundary condition and simplify (omitting the dependencies of $\lambda_{1,2}$ and $A_{1,2}$ on s):

$$A_1e^{\lambda_1 L} + (\hat{u} - A_1)e^{\lambda_2 L} = -\frac{b}{B_0 s}(A_1\lambda_1 e^{\lambda_1 L} + (\hat{u} - A_1)\lambda_2 e^{\lambda_2 L}$$
$$A_1(e^{\lambda_1 L} - e^{\lambda_2 L}) + \hat{u}e^{\lambda_2 L} = A_1(-\frac{b}{B_0 s}\lambda_1 e^{\lambda_1 L} + \frac{b}{B_0 s}\lambda_2 e^{\lambda_2 L}) - \frac{b}{B_0 s}\hat{u}\lambda_2 e^{\lambda_2 L}$$

Collecting the A_1 terms and dividing gives:

$$A_1 = \frac{\hat{u}(-e^{\lambda_2 L} - \frac{b}{B_0 s}\lambda_2 e^{\lambda_2 L})}{\frac{b}{B_0 s}(\lambda_1 e^{\lambda_1 L} - \lambda_2 e^{\lambda_2 L}) + e^{\lambda_1 L} - e^{\lambda_2 L}}$$

Finally, we insert the results for $\lambda_{1,2}$ from Problem 1b (ii), equation (7). We take $\lambda_1 = \alpha - \gamma$ and $\lambda_2 = \alpha + \gamma$:

$$A_1 = \frac{\hat{u}e^{\alpha L}e^{\gamma L}(1 + (\alpha + \gamma)\frac{b}{B_0 s})}{\frac{b}{B_0 s}((\alpha + \gamma)e^{\gamma L}) - (\alpha - \gamma)e^{-\gamma L}) + e^{\gamma L} - e^{-\gamma L})e^{\alpha L}}$$
$$A_1 = \frac{\hat{u}(1 + (\alpha + \gamma)\frac{b}{B_0 s})e^{\gamma L}}{\frac{b}{B_0 s}((\alpha + \gamma)e^{\gamma L} - (\alpha - \gamma)e^{-\gamma L}) + e^{\gamma L} - e^{\gamma L}} \tag{11}$$

An identical procedure can be carried out to find $A_2(s)$ by using $A_1(s) = \hat{u} - A_1(s)$ in the beginning of this section instead of $A_2(s) = \hat{u}(s) - A_1(s)$. The algebra is omitted here but the result follows:

$$A_2 = -\frac{\hat{u}(1 + (\alpha - \gamma)\frac{b}{B_0 s})e^{-\gamma L}}{\frac{b}{B_0 s}((\alpha + \gamma)e^{\gamma L} - (\alpha - \gamma)e^{-\gamma L}) + e^{\gamma L} - e^{\gamma L}} \tag{12}$$

Problem 1e

Conclude with an equation relating $\hat{q}(x, s)$ and $\hat{u}(s)$. Use the hyperbolic sine and cosine functions ($sinh(x) = \frac{e^x - e^{-x}}{2}$ and $cosh(x) = \frac{e^x + e^{-x}}{2}$).

We begin by plugging the equations for $A_{1,2}(s)$ ((11) and (12)) into the equation for $\hat{q}(x, s)$ (equation (8)):

$$\hat{q}(x,s) = \frac{\hat{u}(1+(\alpha+\gamma)\frac{b}{B_0 s})e^{\gamma L}}{\frac{b}{B_0 s}((\alpha+\gamma)e^{\gamma L}-(\alpha-\gamma)e^{-\gamma L})+e^{\gamma L}-e^{-\gamma L}}e^{(\alpha-\gamma)x}$$
$$-\frac{\hat{u}(1+(\alpha-\gamma)\frac{b}{B_0 s})e^{-\gamma L}}{\frac{b}{B_0 s}((\alpha+\gamma)e^{\gamma L}-(\alpha-\gamma)e^{-\gamma L})+e^{\gamma L}-e^{-\gamma L}}e^{(\alpha-\gamma)x}$$

We now collect terms and simplify:

$$\hat{q}(x,s) = \hat{u}e^{\alpha x}(\frac{(1+(\alpha+\gamma)\frac{b}{B_0 s})e^{\gamma L}e^{-\gamma x}}{\frac{b}{B_0 s}((\alpha+\gamma)e^{\gamma L}-(\alpha-\gamma)e^{-\gamma L})+e^{\gamma L}-e^{-\gamma L}}$$
$$-\frac{(1+(\alpha-\gamma)\frac{b}{B_0 s})e^{-\gamma L}e^{\gamma x}}{\frac{b}{B_0 s}((\alpha+\gamma)e^{\gamma L}-(\alpha-\gamma)e^{-\gamma L})+e^{\gamma L}-e^{-\gamma L}})$$

$$\hat{q}(x,s) = \hat{u}e^{\alpha x}(\frac{(1+(\alpha+\gamma)\frac{b}{B_0 s})e^{\gamma(L-x)}}{\frac{b}{B_0 s}(\alpha e^{\gamma K}+\gamma e^{\gamma L}-\alpha e^{-\gamma L}+\gamma e^{-\gamma L})+e^{\gamma L}-e^{-\gamma L}}$$
$$-\frac{(1+(\alpha+\gamma)\frac{b}{B_0 s})e^{\gamma(L-x)}}{\frac{b}{B_0 s}(\alpha e^{\gamma K}+\gamma e^{\gamma L}-\alpha e^{-\gamma L}+\gamma e^{-\gamma L})+e^{\gamma L}-e^{-\gamma L}}$$

$$\hat{q}(x,s) = \hat{u}e^{\alpha x}(\frac{e^{\gamma(L-x)}+\alpha\frac{b}{B_0 s}e^{\gamma(L-x)}+\gamma\frac{b}{B_0 s}e^{\gamma(L-x)}}{\frac{b}{B_0 s}(2\alpha sinh(\gamma L)+2\gamma cosh(\gamma L))+2sinh(\gamma L)}$$
$$-\frac{e^{-\gamma(L-x)}+\alpha\frac{b}{B_0 s}e^{-\gamma(L-x)}-\gamma\frac{b}{B_0 s}e^{-\gamma(L-x)}}{\frac{b}{B_0 s}(2\alpha sinh(\gamma L)+2\gamma cosh(\gamma L))+2sinh(\gamma L)}$$

Simplification continues as the two terms are combined until a satisfactory conclusion is made:

$$\hat{q}(x,s) = \hat{u}e^{\alpha x}\frac{e^{\gamma(L-x)}+\alpha\frac{b}{B_0 s}e^{\gamma(L-x)}+\gamma\frac{b}{B_0 s}e^{\gamma(L-x)}-e^{-\gamma(L-x)}-\alpha\frac{b}{B_0 s}e^{-\gamma(L-x)}+\gamma\frac{b}{B_0 s}e^{-\gamma(L-x)}}{\frac{b}{B_0 s}(2\alpha sinh(\gamma L)+2\gamma cosh(\gamma L))+2sinh(\gamma L)}$$

$$\hat{q}(x,s) = \hat{u}e^{\alpha x}\frac{2sinh(\gamma(L-x))+2\alpha\frac{b}{B_0 s}sinh(\gamma(L-x))+2\gamma\frac{b}{B_0 s}cosh(\gamma(L-x))}{\frac{b}{B_0 s}(2\alpha sinh(\gamma L)+2\gamma cosh(\gamma L))+2sinh(\gamma L)}$$

$$\hat{q}(x,s) = \hat{u}e^{\alpha x}\frac{(B_0 s)sinh(\gamma(L-x))+(\alpha b)sinh(\gamma(L-x))+(\gamma b)cosh(\gamma(L-x))}{\frac{b}{B_0 s}(2\alpha sinh(\gamma L)+2\gamma cosh(\gamma L))+2sinh(\gamma L)}$$

$$\hat{q}(x,s) = \hat{u}e^{\alpha x}\frac{(B_0 s+\alpha b)sinh(\gamma(L-x))+(\gamma b)cosh(\gamma(L-x))}{(B_0 s+\alpha b)sinh(\gamma L)+(\gamma b)cosh(\gamma L)} \tag{13}$$

Problem 2a

Show that $\frac{\hat{q}(x,s)}{\hat{y}(s)} = \frac{P(x,s)}{P(L,s)}$ *and deduce a given relationship between* $\hat{q}(x,s)$ *and* $\hat{y}(s)$.

The power series expansions of the hyperbolic sine and cosine will be used in this problem and were given in the problem statement. They are reproduced below:

$$sinh(x) = \sum_{i=0}^{\infty} \frac{x^{2i+1}}{(2i+1)!} \tag{14}$$

$$cosh(x) = \sum_{i=0}^{\infty} \frac{x^{2i}}{(2i)!} \tag{15}$$

We use $P(x,s)$ as defined in the problem statement of Problem 1e. It is the numerator of $\hat{q}(x,s)$ (excluding $\hat{u}(s)$). Additionally, $y(t)$ is given as $q(L,t)$. We reference the definition of the Laplace transform in equation (4), plugging $y(t)$ into it, following the same procedure as was done previously in Problem 1a with equation (1), as well as in Problem 1c with equation (2):

$$\hat{y}(s) = \hat{q}(L,s) \tag{16}$$

We begin by showing that $\frac{\hat{q}(x,s)}{\hat{y}(s)} = \frac{P(x,s)}{P(L,s)}$. We make use of the properties $sinh(0) = 0$ and $cosh(0) = 1$. The left hand side follows:

$$\frac{\hat{q}(x,s)}{\hat{q}(L,s)} = \frac{\hat{u}e^{\alpha x}\frac{(B_0 s+\alpha b)sinh(\gamma(L-x))+(\gamma b)cosh(\gamma(L-x))}{(B_0 s+\alpha b)sinh(\gamma L)+(\gamma b)cosh(\gamma L)}}{\hat{u}e^{\alpha L}\frac{\gamma b}{(B_0 s+\alpha b)sinh(\gamma L)+(\gamma b)cosh(\gamma L)}}$$

$$\frac{\hat{q}(x,s)}{\hat{q}(L,s)} = \frac{e^{\alpha x}}{e^{\alpha L}}\frac{(B_0 s+\alpha b)sinh(\gamma(L-x))+(\gamma b)cosh(\gamma(L-x))}{\gamma b}$$

$$\frac{\hat{q}(x,s)}{\hat{q}(L,s)} = e^{\alpha(x-L)}(\frac{B_0 s+\alpha b)sinh(\gamma(L-x))}{\gamma b} + cosh(\gamma(L-x))) \tag{17}$$

Similarly, the right hand side is found as follows:

$$\frac{P(x,s)}{P(L,s)} = \frac{e^{\alpha x}((B_0 s+\alpha b)sinh(\gamma(L-x))+(\gamma b)cosh(\gamma(L-x)))}{\gamma b}$$

$$\frac{P(x,s)}{P(L,s)} = e^{\alpha(x-L)}(\frac{B_0 s+\alpha b)sinh(\gamma(L-x))}{\gamma b} + cosh(\gamma(L-x))) \tag{18}$$

Since the right hand sides of equations (17) and (18) match, it has been shown that $\frac{\hat{q}(x,s)}{\hat{y}(s)} = \frac{P(x,s)}{P(L,s)}$ (referencing equation (16) if necessary). We now turn our efforts to deducing an expression relating $\hat{q}(x,s)$

and $\hat{y}(s)$. From above:

$$\hat{q}(x,s) = \frac{P(x,s)}{P(L,s)}\hat{y}(s)$$

$$\hat{q}(x,s) = y(\hat{s})e^{\alpha(x-L)}(\frac{(B_0 s + \alpha b)}{\gamma b}sinh(\gamma(L-x)) + cosh(\gamma(L-x)))$$

Reviewing trigonometric properties, we find the following: $sinh(x-L) = -sinh(L-x)$ and $cosh(x-L) = cosh(L-x)$, keeping in mind that $(x-L) = -(L-x)$. We make use of these properties in order to achieve the desired form:

$$\hat{q}(x,s) = y(\hat{s})e^{\alpha(x-L)}(\frac{-(B_0 s + \alpha b)}{\gamma b}sinh(\gamma(x-L)) + cosh(\gamma(x-L)))$$

Now, making use of equations (14) and (15):

$$\hat{q}(x,s) = e^{\alpha(x-L)}(\frac{-B_0 s - \alpha b}{\gamma b}\sum_{i=0}^{\infty}\frac{(\gamma(x-L))^{2i+1}}{(2i+1)!} + \sum_{i=0}^{\infty}\frac{(\gamma(x-L))^{2i}}{(2i)!})\hat{y}(s)$$

$$\hat{q}(x,s) = e^{\alpha(x-L)}(\frac{-B_0 s}{\gamma b}\sum_{i=0}^{\infty}(\gamma^2)^{i+\frac{1}{2}}\frac{(x-L)^{2i+1}}{(2i+1)!} - \frac{\alpha}{\gamma}\sum_{i=0}^{\infty}(\gamma^2)^{i+\frac{1}{2}}\frac{(x-L)^{2i+1}}{(2i+1)!} + \sum_{i=0}^{\infty}(\gamma^2)^{i}\frac{(x-L)^{2i}}{(2i)!})\hat{y}(s)$$

Next, we reference the definition of γ from Problem 1b (ii), $\gamma(s) = \beta\sqrt{s + \frac{\alpha^2}{\beta^2}}$, to find that:

$$\hat{q}(x,s) = e^{\alpha(x-L)}(\frac{-B_0 s}{\beta b\sqrt{s+\frac{\alpha^2}{\beta^2}}}\sum_{i=0}^{\infty}(s+\frac{\alpha^2}{\beta^2})^{i+\frac{1}{2}}\frac{\beta^{2i+1}(x-L)^{2i+1}}{(2i+1)!} - \frac{\alpha}{\beta\sqrt{s+\frac{\alpha^2}{\beta^2}}}\sum_{i=0}^{\infty}(s+\frac{\alpha^2}{\beta^2})^{i+\frac{1}{2}}\frac{\beta^{2i+1}(x-L)^{2i+1}}{(2i+1)!}$$
$$+ \sum_{i=0}^{\infty}\beta^{2i}(s+\frac{\alpha^2}{\beta^2})^{i}\frac{(x-L)^{2i}}{(2i)!})\hat{y}(s)$$

We then multiply the $\sqrt{s+\frac{\alpha^2}{\beta^2}}$ terms back into the summations:

$$\hat{q}(x,s) = e^{\alpha(x-L)}(\frac{-B_0 s}{\beta b}\sum_{i=0}^{\infty}(s+\frac{\alpha^2}{\beta^2})^{i+1}\frac{\beta^{2i+1}(x-L)^{2i+1}}{(2i+1)!} - \frac{\alpha}{\beta}\sum_{i=0}^{\infty}(s+\frac{\alpha^2}{\beta^2})^{i+1}\frac{\beta^{2i+1}(x-L)^{2i+1}}{(2i+1)!}$$
$$+ \sum_{i=0}^{\infty}\beta^{2i}(s+\frac{\alpha^2}{\beta^2})^{i}\frac{(x-L)^{2i}}{(2i)!})\hat{y}(s)$$

Problem 2a continued on next page. . .

Continuing to simplify:

$$\hat{q}(x,s) = e^{\alpha(x-L)}(\sum_{i=0}^{\infty}\beta^{2i}(\frac{-B_0 s}{b}(s+\frac{\alpha^2}{\beta^2})^{i+1} - \alpha(s+\frac{\alpha^2}{\beta^2})^{i+1})\frac{(x-L)^{2i+1}}{(2i+1)!} + \sum_{i=0}^{\infty}\beta^{2i}(s+\frac{\alpha^2}{\beta^2})^{i}\frac{(x-L)^{2i}}{(2i)!})\hat{y}(s)$$

Finally, after further simplification, including dividing $(s+\frac{\alpha^2}{\beta^2})$ out from within the first summation, we find the desired solution:

$$\hat{q}(x,s) = e^{\alpha(x-L)}(\sum_{i=0}^{\infty}\beta^{2i}((\frac{B_0\alpha^2}{b\beta^2}-\alpha)(s+\frac{\alpha^2}{\beta^2})^{i} - \frac{B_0}{b}(s+\frac{\alpha^2}{\beta^2})^{i+1})\frac{(x-L)^{2i+1}}{(2i+1)!} + \sum_{i=0}^{\infty}\beta^{2i}(s+\frac{\alpha^2}{\beta^2})^{i}\frac{(x-L)^{2i}}{(2i)!})\hat{y}(s) \tag{19}$$

Problem 2b

Map the result from above back into the time domain by using the inverse Laplace transform identity: $\mathcal{L}^{-1}((s+a)^i\hat{g}(s)) = e^{-at}(g(t)e^{at})^i$ *where* a *is a constant and* $g(t)$ *is a time function.*

By examining the formula for $\hat{q}(s)$ in equation (19), we see that $(s+\frac{\alpha^2}{\beta^2})$ is present in every term, so we exercise an artistic choice and begin by trying the inverse Laplace transform with $a = \frac{\alpha^2}{\beta^2}$:

$$\mathcal{L}^{-1}(\hat{q}(x,s)) = q(x,t)$$
$$\mathcal{L}^{-1}((s+a)^i\hat{g}(s)) = e^{-at}(g(t)e^{at})^{(i)} \tag{20}$$

Where the superscript (i) indicates the term number (i.e., $(i)th$) rather than exponentiation. Next, using $k = \frac{B_0\alpha^2}{b\beta^2} - \alpha$ we factor $(s+\frac{\alpha^2}{\beta^2})^i$ out of $\hat{q}(x,t)$ to get $\hat{g}(s)$:

$$\hat{g}(s) = e^{\alpha(x-L)}(\sum_{i=0}^{\infty}\frac{\beta^{2i}(x-L)^{2i}}{(2i)!} + \sum_{i=0}^{\infty}\beta^{2i}(k - \frac{B_0}{b}(s+\frac{\alpha^2}{\beta^2}))\frac{(x-L)^{2i+1}}{(2i+1)!})\hat{y}(s)$$

We now apply $\hat{g}(s)$ as defined above and our intuition-guided choice of $a = \frac{\alpha^2}{\beta^2}$ to equation (20). Note that the terms resulting from $\hat{y}(x)$ will carry a superscript order (i) as a result of their transformation from the Laplace domain back to the time domain. Likewise, the e^{at} terms will also carry an order (i). This is prescribed by equation (20).

$$q(x,t) = e^{-\frac{\alpha^2}{\beta^2}t} e^{\alpha(x-L)} (\sum_{i=0}^{\infty} (e^{\frac{\alpha^2}{\beta^2}t})^{(i)} \frac{\beta^{2i}(x-L)^{2i}}{(2i)!} y^{(i)} + k \sum_{i=0}^{\infty} (e^{\frac{\alpha^2}{\beta^2}t})^{(i)} \frac{\beta^{2i}(x-L)^{2i+1}}{(2i+1)!} y^{(i)}$$
$$+ \sum_{i=0}^{\infty} (e^{\frac{\alpha^2}{\beta^2}t})^{(i)} \beta^{2i} \frac{-B_0}{b} (s + \frac{\alpha^2}{\beta^2}) \frac{(x-L)^{2i+1}}{(2i+1)!}) y^{(i)}$$

As can be seen, there is an extra $(s+a)$ term in the third summation. In order to remove it, we use the $(i+1)th$ ordered term there instead of the $(i)th$ and continue to simplify to find the desired result:

$$q(x,t) = e^{-\frac{\alpha^2}{\beta^2}t + \alpha(x-L)} (\sum_{i=0}^{\infty} (e^{\frac{\alpha^2}{\beta^2}t} y)^{(i)} \frac{\beta^{2i}(x-L)^{2i}}{(2i)!} + k \sum_{i=0}^{\infty} (e^{\frac{\alpha^2}{\beta^2}t} y)^{(i)} \frac{\beta^{2i}(x-L)^{2i+1}}{(2i+1)!}$$
$$- \frac{B_0}{b} \sum_{i=0}^{\infty} (e^{\frac{\alpha^2}{\beta^2}t} y)^{(i+1)} \frac{\beta^{2i}(x-L)^{2i+1}}{(2i+1)!}) \quad (21)$$

Problem 3a (Bonus)

Implement and plot the `bump.m` *function.*

Listing 1: Script for the bump function

```
function gevrey = bump(t)
%Bump function written for Problem 3
%CE291F Take Home Midterm. Spring 2009.
%Eric Mai.
%April 8, 2009.

phit = @(x) exp(-1./(x.*(1-x)));
denomf = @(x) exp(-1./(x.*(1-x)));

denom=quadgk(denomf,0,1);

oneSecond=(10000-0.4*3600)/(5*3600);
gevrey(1:5)=0;

for ii=[6:length(t)]
    if t(ii)<oneSecond
        if isnan((quadgk(phit,0,t(ii))/denom)) || isinf((quad(phit,0,t(ii))/denom))
            gevrey(ii)=1;
        else gevrey(ii)=(quadgk(phit,0,t(ii))/denom);
        end
    else
        gevrey(ii)=1;
    end
end

return;
```

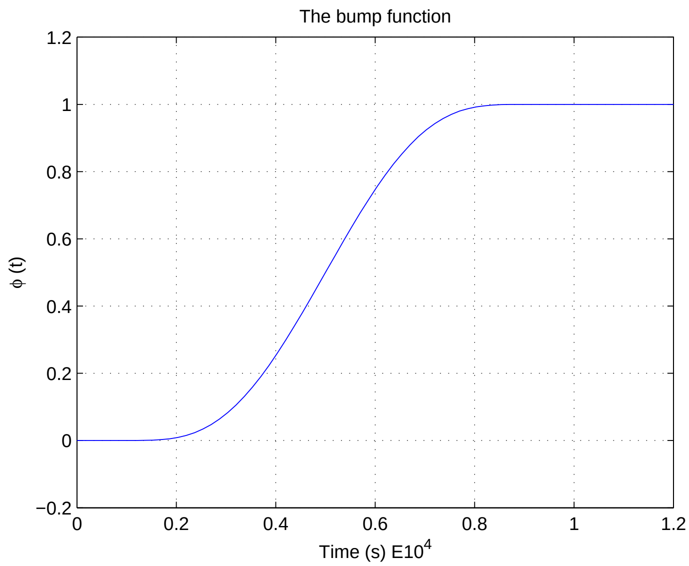

Figure 1: The bump function in Problem 3a.

Problem 3b (Bonus)

Run `exam.m` *and comment on the results.*

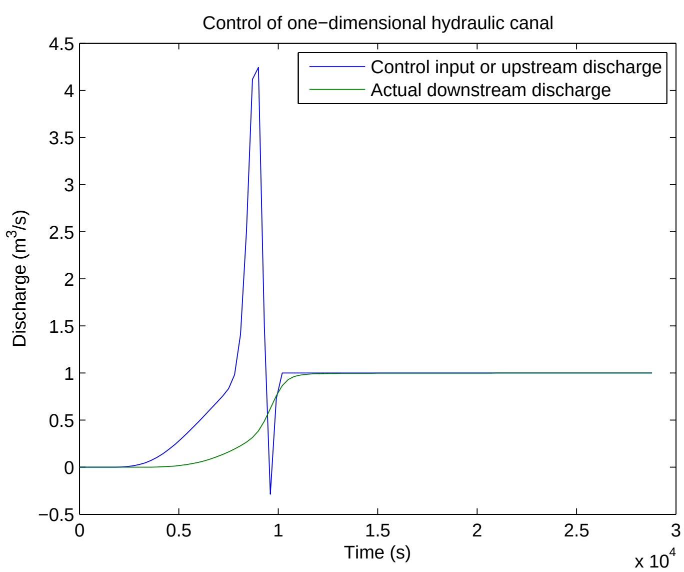

Figure 2: The results of running `exam.m` in Problem 3b.

i) Comment on the presence or absence of a delay between the input and output:

Yes, there is a delay between the input and output. It can be seen that the input (the upstream discharge) begins rising around t=0.5×10^4 seconds while the output discharge does not begin its increase until close to t=1×10^4 seconds. Additionally, the discharge begins to slow its increase shortly after the time when the control input drops dramatically. Incidentally, this control is a vivid example of a "bang bang" control mechanism.

ii) Comment on the diffusion you see:

The diffusion can be seen in Figure 2 as, even though the control remains constant after t=1×10^4 seconds, the actual downstream discharge continues to change. In a sense, the change in the discharge is "diffusing" beyond this threshold, due to the delay.

Problem 3c (Bonus)

Modify the bump function and repeat the steps of 3a and 3b.

Listing 2: Script for the modified bump function: A decaying sine

```
function gevrey = bump2(t)
%Alternate bump function written for Problem 3
%CE291F Take Home Midterm. Spring 2009.
%Eric Mai.
%April 8, 2009.

oneandhalfSeconds=(15000-0.4*3600)/(5*3600);

for ii=[1:length(t)]
    if t(ii)<oneandhalfSeconds
        gevrey(ii)=exp(-t(ii))*2.5*sin(5*pi()*t(ii));
    else
        gevrey(ii)=1;
    end
end

return;
```

It can be seen that the control is fairly robust as it is able to achieve a downstream discharge which looks roughly like some modification of a trigonometric function. The comments above in Problem 3b regarding the delay between the input and the output, as well as the diffusion, are even more valid in this case. It can be clearly seen that the period of the control combined with the input/output delay cause the system to be letting the water in when small amounts of water are discharging and vice versa.

Problem 4 (Bonus)

Prove the convergence of the three summations in equation (21)

Following the problem statement, we assume that $y(t)$ is a Gevrey function of order $\psi \geq 0$. We assume that this order is preserved under the transformation $e^{\frac{\alpha^2}{\beta^2}t}$, i.e., $e^{\frac{\alpha^2}{\beta^2}t}y$ and $y(t)$ have the same order ψ.

It is also given that for a Gevrey function of order $\alpha \geq 0$ some m, l, and α are guaranteed to exist such that for any $n \in \mathbb{N}$, the following expression holds:

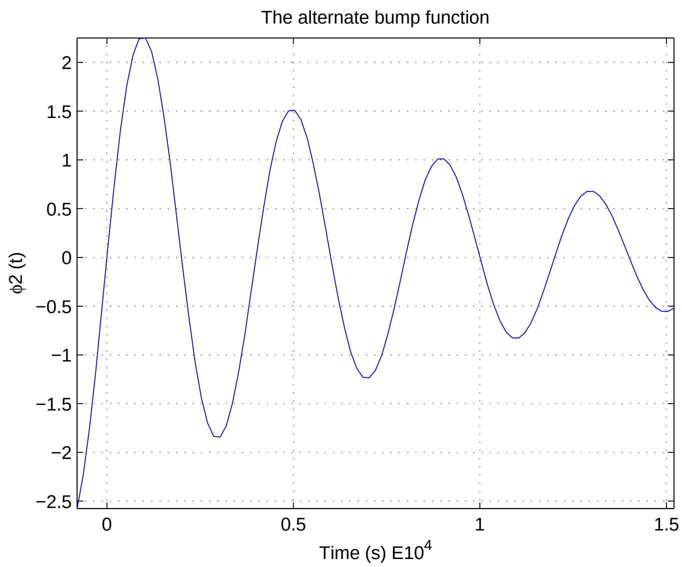

Figure 3: The modified bump function in Problem 3c.

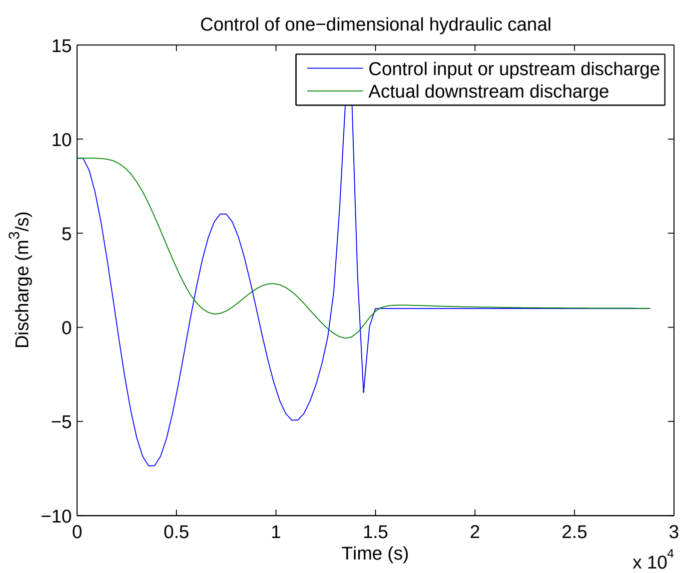

Figure 4: The results of running `exam.m` in Problem 3c.

$$\exists\ m, l \in \mathbb{R}\ \exists\ \alpha \geq 0\ \forall n \in \mathbb{N} \sup_{t\in[0,T]} |f_0^{(n)}(t)| < m\frac{(n!)^\alpha}{l^n}$$

Finally, Cauchy-Hadamard gives an expression for the radius of convergence of a series:

$$\text{The radius of convergence of the series } \sum_{i=0}^{\infty} a_n x^n \text{ is } \frac{1}{\lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}}$$

The approach is to use $m\frac{(n!)^\alpha}{l^n}$ for $|f_0^(n)(t)|$ because it represents the most divergent scenario. If $\sum_{i=0}^{\infty} a_n x^n \leq \frac{1}{\lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}}$ is satisfied under this condition, then it is safe to say that the series converges. Finally, if all three summations in equation (21) converge under this test, then it can be said that a useful solution for $q(x,t)$ exists.

We take $T_{1,2,3}(x,t)$ to be the three summations making up $\hat{q}(x,s)$ defined as in the problem statement of Problem 2b. It can be seen that the orders of $T_1(x,t)$ and $T_2(x,t)$ are both i while the order of $T_3(x,t)$ is $(i+1)$.

Since the problem only asks for a discussion of the convergence of one of the $T(x,t)$ terms, $T_1(x,t)$ is chosen and examined below.

$$T_1(x,t) = \sum_{i=0}^{\infty} (e^{\frac{\alpha^2}{\beta^2}t} y)^{(i)} \frac{\beta^{2i}(x-L)^{2i}}{(2i)!}$$

For $a = \frac{\alpha^2}{\beta^2}$ as before, and order $n = i$, as can be seen above, the radius of convergence of the series is given by Cauchy-Hadamard to be:

$$\frac{1}{\lim_{i\to\infty} \sup |\frac{\alpha^2}{\beta^2}{}_i|^{\frac{1}{i}}}$$

Furthermore, we know that there are values m, l, and α which can satisfy $\sup_{t\in[0,T]} |A_{1,0}^{(i)}(t)| < m\frac{(i!)^\alpha}{l^i}$.

Further manipulation is omitted; we leave it to the reader to conclude that $A_1(x,t)$ is within the Cauchy-Hadamard radius of convergence for all values of x and t.